$$S_\varkappa = -\varkappa \frac{dT}{dr} = \frac{\varkappa a}{r^2}(T_a - T_\infty)\left[1 - \frac{1}{3\mu^2\alpha^2} + \frac{E_4(\alpha r) + \alpha r E_3(\alpha r)}{\mu^2\alpha^2}\right] \qquad (7.4)$$

At the large distances where the diffusion approximation is valid for $\mu^2\alpha^2 \gg 1$ we can use the expression for $S_\varkappa$ at $\alpha r \gg 1$ to obtain the total energy flux density

$$S_0 = S_\varkappa\left(1 + \frac{1}{3\mu^2\alpha^2}\right) = \frac{\varkappa a}{r^2}(T_a - T_\infty) \qquad (7.5)$$

This implies that the radiant energy flux density over the whole space is

$$S = \frac{16\sigma a T_\infty{}^3}{\alpha r^2}\left[\frac{1}{3} - E_4(\alpha r) - \alpha r E_3(\alpha r)\right](T_a - T_\infty) \qquad (7.6)$$

We see here that in the region $\alpha r \ll 1$ the influence of the molecular mode of the energy transfer is dominant and the radiant energy flux is practically absent. When $\alpha r \gg 1$, the flux tends exponentially to the limit defined by the radiant tranfer approximation, and is a small quantity of the order of $(\mu\alpha)^{-2}S_\varkappa$.

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INFINITE ELASTIC LAYER AND HALF-SPACE UNDER THE ACTION OF A RING-SHAPED DIE

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The problem of pressure due to an axisymmetric ring-shaped die on an elastic half-space and layer was examined in [1 and 2]. In these papers the boundary value problem of the theory of elasticity is reduced to a linear integral equation of the second kind with a kernel given by a set of infinite measure. In [3] the problem of pressure due to a ring-shaped die on an elastic layer is reduced to a Fredholm integral equation of the second kind by means of approximate substitution of the kernel of the integral equation of the first kind. Normal stresses under the die are expressed through the derivative of the solution of this equation. In papers [4 and 5] the problem of pressure due to an axisymmetric ring-shaped die on a half-space was solved by approximate methods.

In this paper the axisymmetric problem of pressure due to a ring-shaped die on an infinite elastic layer and half-space is solved and also the problem of torsion of the elastic layer and half-space under the influence of a coupled rigid die. In addition to the die, the half-space and layer are under the influence of a steady-state temperature field. The solution of boundary value problems are presented in the form of integrals which contain an unknown function which is determined by triple integral equations. In this connection problems related to the pressure of the die are reduced to one type of triple integral equations and problems of torsion to another type of triple equations. Solutions of the first as well as the second type of triple integral equations are presented in the form of integrals containing two auxiliary functions which, being given on adjacent intervals, form one discontinuous function. This function is found from the integral equation of Fredholm of the second kind. The law of stress distribution under the die is found. In this connection the indicated stresses are expressed directly through solutions of the equation of Fredholm.

1. Equations of Duhamel-Neumann for the case of axisymmetric thermoelastic deformation of a body have the form

$$(1-\sigma)\frac{\partial\theta}{\partial r}+(1-2\sigma)\frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)=\alpha(1+\sigma)\frac{\partial T}{\partial r}$$
$$(1-\sigma)\frac{\partial\theta}{\partial z}-(1-2\sigma)\frac{1}{2r}\frac{\partial}{\partial r}\left[r\left(\frac{\partial u_r}{\partial z}-\frac{\partial u_z}{\partial r}\right)\right]=\alpha(1+\sigma)\frac{\partial T}{\partial z} \tag{1.1}$$

Here θ is the bulk deformation, σ is the Poisson ratio, T is the temperature of the body, α is the constant coefficient of linear expansion due to temperature, u_r, u_z are projections of the displacement vector in a cylindrical system of coordinates r, φ, z. Solution of Eqs. (1.1) is presented in the form [6]

$$u_r=\frac{-1}{4(1-\sigma)}\frac{\partial}{\partial r}(z\delta_3+\delta_1),\qquad u_z=\delta_3-\frac{1}{4(1-\sigma)}\frac{\partial}{\partial z}(z\delta_3+\delta_2) \tag{1.2}$$

where δ_3 is an arbitrary harmonic function, δ_1 and δ_2 are harmonic functions related to temperature by the relationship

$$\frac{\partial^2(\delta_2-\delta_1)}{\partial z^2}=8\alpha(\sigma^2-1)T \tag{1.3}$$

where the temperature T is assumed to be a harmonic function. The components of the stress tensor corresponding to Eqs. (1.2) are as follows:

$$\sigma_z=\frac{-G}{2(1-\sigma)}\frac{\partial^2}{\partial z^2}\left[z\delta_3+\delta_2+\frac{\sigma}{1-2\sigma}(\delta_2-\delta_1)\right]+\frac{G(2-\sigma)}{1-\sigma}\frac{\partial\delta_3}{\partial z}-\beta T$$
$$\sigma_r=\frac{-G}{2(1-\sigma)}\left[\frac{\partial^2}{\partial r^2}(z\delta_3+\delta_1)+\frac{\sigma}{1-2\sigma}\frac{\partial^2(\delta_2-\delta_1)}{\partial z^2}-2\sigma\frac{\partial\delta_3}{\partial z}\right]-\beta T$$
$$\sigma_\varphi=\frac{-G}{2(1-\sigma)}\left[\frac{1}{r}\frac{\partial}{\partial r}(z\delta_3+\delta_1)+\frac{\sigma}{1-2\sigma}\frac{\partial^2(\delta_2-\delta_1)}{\partial z^2}-2\sigma\frac{\partial\delta_3}{\partial z}\right]-\beta T \tag{1.4}$$
$$\tau_{rz}=\frac{-G}{4(1-\sigma)}\frac{\partial^2}{\partial r\,\partial z}(2z\delta_3+\delta_1+\delta_2)+G\frac{\partial\delta_3}{\partial r}\qquad\left(\beta=\frac{2\alpha(1+\sigma)G}{1-2\sigma}\right)$$

Here G is the shear modulus.

Equations (1.2) and (1.4) are used for the solution of contact problems examined below for the infinite elastic layer and half-space. For the case of the infinite layer the harmonic functions entering into these equations are taken in the form

$$\delta_1=\int_0^{+\infty}[A_3(\lambda)\operatorname{sh}\lambda z+A_4(\lambda)\operatorname{ch}\lambda z]J_0(\lambda r)\,d\lambda$$
$$\delta_2=\int_0^{+\infty}[A_1(\lambda)\operatorname{sh}\lambda z+A_2(\lambda)\operatorname{ch}\lambda z]J_0(\lambda r)\,d\lambda \tag{1.5}$$

$$\delta_3 = \int_0^{+\infty} [A_5(\lambda)\operatorname{sh}\lambda z + A_6(\lambda)\operatorname{ch}\lambda z] J_0(\lambda r)\, d\lambda$$

where $A_1(\lambda)$, $A_2(\lambda), \ldots, A_6(\lambda)$ are unknown functions which are determined from the condition of relationship (1.3) and from boundary conditions.

In the following we shall assume the temperature $T(r, z)$ as a given function, to be a solution of some boundary value problem for the Laplace equation for the layer $-l \leqslant z \leqslant l$, $0 \leqslant r < +\infty$. Therefore we consider it to be representable by the improper integral

$$T(r, z) = \int_0^{+\infty} [C_1(\lambda)\operatorname{sh}\lambda z + C_2(\lambda)\operatorname{ch}\lambda z] J_0(\lambda z)\, d\lambda \qquad (1.6)$$

where functions $C_1(\lambda)$ and $C_2(\lambda)$ are determined from boundary conditions, i.e. from the temperature conditions on boundary planes $z = \pm l$. The condition of relationship (1.3) gives the following dependence between unknown functions:

$$A_1(\lambda) = A_3(\lambda) - \frac{\gamma}{\lambda^2} C_1(\lambda), \quad A_2(\lambda) = A_4(\lambda) - \frac{\gamma}{\lambda^2} C_2(\lambda) \qquad (1.7)$$

$$\gamma = \frac{4\beta}{G}(1 - 2\sigma)(1 - \sigma)$$

In the case of the half-space we take the harmonic functions entering into Eqs. (1.2) in the form

$$\delta_1 = \int_0^{+\infty} A_1(\lambda) e^{-\lambda z} J_0(\lambda r)\, d\lambda, \quad \delta_2 = \int_0^{+\infty} A_2(\lambda) e^{-\lambda z} J_0(\lambda r)\, d\lambda$$

$$\delta_3 = \int_0^{+\infty} A_3(\lambda) e^{-\lambda z} J_0(\lambda r)\, d\lambda \qquad (1.8)$$

The temperature is considered to be representable by the improper integral

$$T(r, z) = \int_0^{+\infty} C(\lambda) e^{-\lambda z} J_0(\lambda r)\, d\lambda \qquad (1.9)$$

where the function $C(\lambda)$ is determined from the temperature conditions on the boundary of the half-space $z \geqslant 0$. The condition of the relationship (1.3) gives the following dependence between the unknown functions $A_1(\lambda)$ and $A_2(\lambda)$:

$$A_1(\lambda) = A_2(\lambda) + \frac{\gamma}{\lambda^2} C(\lambda) \qquad (1.10)$$

2. The boundary value problems examined below are reduced to the following triple integral equations (*):

$$\int_0^{+\infty} \lambda A(\lambda) J_0(\lambda r)\, d\lambda = F_1(r) \qquad (0 \leqslant r < a) \qquad (2.1)$$

$$\int_0^{+\infty} A(\lambda)[1 - g(\lambda)] J_0(\lambda r)\, d\lambda = F_2(r) \qquad (a \leqslant r \leqslant R) \qquad (2.2)$$

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*) The integral equations to be solved here were examined by Cooke [7] and Tranter [8] for $q(\lambda) = 0$ and $F_1(r) = 0$. The kernel of the integral equation of the second kind obtained by Cooke has a nonsummable square. Therefore the question of existence of solution of equations remained open. In the paper of Tranter, on the other hand, triple integral equations are reduced to the equivalent problem of dual series equations.

$$\int_0^{+\infty} \lambda A(\lambda) J_0(\lambda r)\, d\lambda = 0 \qquad (R < r < +\infty) \tag{2.3}$$

Here $A(\lambda)$ is an unknown function, while $g(\lambda)$, $F_1(r)$ and $F_2(r)$ are given functions. We assume that function $g(\lambda)$ is continuous, while $\lambda^3 g(\lambda)$ is absolutely integrable over $[0, +\infty)$; the function $F_1(r)$ is such that the integral

$$(a^2 - t^2)^{1/4} \int_0^t \frac{r F_1(r)\, dr}{\sqrt{t^2 - r^2}} \qquad (0 \leqslant t \leqslant a)$$

becomes zero for $t = 0$ and turns out to be a function which is square-summable.

We are looking for the solution of triple integral equations in the form

$$A(\lambda) = \int_0^a \varphi_1(t) \sin \lambda t\, dt + \int_a^R \varphi_2(t) \cos \lambda t\, dt + \int_R^{+\infty} \varphi_3(t) \cos \lambda t\, dt \tag{2.4}$$

where it is assumed that

$$\lim_{t \to a} \varphi_1(t)(t - a) = \lim_{t \to a} \varphi_2(t)(t - a) = 0, \quad \varphi_1(0) = 0, \quad \varphi_3(+\infty) = 0 \tag{2.5}$$

Here $\varphi_1(t)$, $\varphi_2(t)$ and $\varphi_3(t)$ are unknown functions which must be found by substitution of function (2.4) into Eqs. (2.1)–(2.3). Completing integration by parts we obtain

$$A(\lambda) = \frac{1}{\lambda} \left\{ \int_0^a \varphi_1'(t) [\cos \lambda t - \cos \lambda a]\, dt + \varphi_2(R) [\sin \lambda R - \sin \lambda a] - \right.$$

$$\left. - \int_a^R \varphi_2'(t) [\sin \lambda t - \sin \lambda a]\, dt - \varphi_3(R) \sin \lambda R - \int_R^{+\infty} \varphi_3'(t) \sin \lambda t\, dt \right\} \tag{2.6}$$

Substituting function (2.6) into Eqs. (2.1) and (2.3) and utilizing integrals

$$\int_0^{+\infty} J_0(\lambda r) \sin \lambda t\, d\lambda = \begin{cases} 0 & \text{for } t < r \\ (t^2 - r^2)^{-1/2} & \text{for } t > r \end{cases}$$

$$\int_0^{+\infty} J_0(\lambda r) \cos \lambda t\, d\lambda = \begin{cases} (r^2 - t^2)^{-1/2} & \text{for } t < r \\ 0 & \text{for } t > r \end{cases} \tag{2.7}$$

we obtain

$$\int_0^r \frac{\varphi_1'(t)\, dt}{\sqrt{r^2 - t^2}} = -\xi(r) + F_1(r) \qquad (0 \leqslant r < a)$$

$$\int_r^{+\infty} \frac{\varphi_3'(t)\, dt}{\sqrt{t^2 - r^2}} = \int_0^a s(r, t) \varphi_1'(t)\, dt \qquad (R < t < +\infty)$$

$$\xi(r) = \frac{\varphi_2(R)}{\sqrt{R^2 - r^2}} - \frac{\varphi_2(R)}{\sqrt{a^2 - r^2}} - \frac{\varphi_3(R)}{\sqrt{R^2 - r^2}} - \tag{2.8}$$

$$- \int_a^R \varphi_2'(t) \left[\frac{1}{\sqrt{t^2 - r^2}} - \frac{1}{\sqrt{a^2 - r^2}} \right] dt - \int_R^{+\infty} \frac{\varphi_3'(t)\, dt}{\sqrt{t^2 - r^2}}$$

$$s(r, t) = \frac{1}{\sqrt{r^2 - t^2}} - \frac{1}{\sqrt{r^2 - a^2}}$$

Solutions of Abel's equations (2.8) have the form

$$\varphi_1'(t)=\frac{2}{\pi}\frac{d}{dt}\int_0^t\frac{r\,[F_1(r)-\xi(r)]}{\sqrt{t^2-r^2}}\,dr \qquad (0\leqslant t<a)$$

$$\varphi_2'(t)=-\frac{2}{\pi}\frac{d}{dt}\int_0^a N_0(t,\tau)\,\varphi_1'(\tau)\,d\tau \qquad (R<t<+\infty)$$

$$N_0(t,\tau)=\int_t^{+\infty}\frac{rs\,(r,\tau)\,dr}{\sqrt{r^2-t^2}}=\frac{1}{2}\ln\frac{t^2-a^2}{t^2-\tau^2} \qquad \begin{pmatrix}R<t<+\infty\\ 0\leqslant\tau\leqslant a\end{pmatrix}$$

From this

$$\varphi_1(t)=\frac{2}{\pi}\int_0^t\frac{r\,[F_1(r)-\xi(r)]}{\sqrt{t^2-r^2}}\,dr+c_1,\quad \varphi_2(t)=-\frac{2}{\pi}\int_0^a N_0(t,\tau)\,\varphi_1'(\tau)\,d\tau+c_2$$

or, computing the integrals in the right parts we obtain

$$\varphi_1(t)=-\frac{2}{\pi}\int_a^R\frac{t\varphi_2(\tau)\,d\tau}{t^2-\tau^2}-\frac{2}{\pi}\int_R^{+\infty}\frac{t\varphi_2(\tau)\,d\tau}{t^2-\tau^2}+\chi_{11}(t) \quad (0\leqslant t<a) \tag{2.9}$$

$$\varphi_2(t)=-\frac{2}{\pi}\int_0^a\frac{\tau\varphi_1(\tau)\,d\tau}{\tau^2-t^2} \qquad (R<t<+\infty) \tag{2.10}$$

$$\chi_{11}(t)=\frac{2}{\pi}\int_0^t\frac{rF_1(r)\,dr}{\sqrt{t^2-r^2}} \tag{2.11}$$

where the constants c_1 and c_2 are equal to zero by virtue of conditions (2.5).

We rewrite Eq. (2.2) as follows:

$$\int_0^{+\infty}A(\lambda)\,J_0(\lambda r)\,d\lambda=F_{21}(r) \qquad (a\leqslant r\leqslant R) \tag{2.12}$$

$$F_{21}(r)=\int_0^{+\infty}g(\lambda)\,A(\lambda)\,J_0(\lambda r)\,d\lambda+F_2(r) \tag{2.13}$$

Substituting function (2.4) into Eqs. (2.12) and (2.13) and utilizing integrals (2.7) and Expressions (2.10) we obtain

$$\int_a^R\frac{\varphi_2(t)\,dt}{\sqrt{r^2-t^2}}=F_{21}(r) \qquad (a\leqslant r\leqslant R) \tag{2.14}$$

$$F_{21}(r)=-\int_0^a\varphi_1(\tau)\,M_1(\tau,r)\,d\tau+\int_a^R\varphi_2(\tau)\,M_2(\tau,r)\,d\tau+F_2(r) \tag{2.15}$$

Here

$$\begin{gathered}M_1(\tau,r)=\tau\int_0^{+\infty}g(\lambda)\,N(\lambda,\tau)\,J_0(\lambda r)\,d\lambda-\int_0^{+\infty}g(\lambda)\,J_0(\lambda r)\sin\lambda\tau\,d\lambda\\ M_2(\tau,r)=\int_0^{+\infty}g(\lambda)\,J_0(\lambda r)\cos\lambda\tau\,d\lambda,\quad N(\lambda,\tau)=\frac{2}{\pi}\int_R^{+\infty}\frac{\cos\lambda t\,dt}{\tau^2-t^2}\end{gathered} \tag{2.16}$$

Solution of Abel's equation (2.14) has the form

$$\varphi_2(t) = \frac{2tF_{21}(a)}{\pi\sqrt{t^2-a^2}} + \frac{2}{\pi}\int_a^t \frac{tF_{21}'(r)\,dr}{\sqrt{t^2-r^2}} \quad (a \leqslant t \leqslant R) \tag{2.17}$$

Equation (2. 9) is transformed substituting into it Expression $\varphi_2(t)$ from (2. 17) and $\varphi_2(t)$ from (2. 10).

As a result of transformations, the indicated equation assumes the form

$$\varphi_1(t) = \int_0^a K_{11}^{(1)}(t, x)\varphi_1(x)\,dx + \frac{4tF_{21}(a)}{\pi^2\sqrt{a^2-t^2}}\operatorname{arc\,tg}\left(\frac{R^2-a^2}{a^2-t^2}\right)^{1/2} +$$

$$+ \frac{4t}{\pi^2}\int_a^R \frac{F_{21}'(r)}{\sqrt{r^2-t^2}}\operatorname{arc\,tg}\left(\frac{R^2-r^2}{r^2-t^2}\right)^{1/2} dr + \chi_{11}(t) \tag{2.18}$$

$$K_{11}^{(1)}(t, x) = \frac{2}{\pi^2}\left[x\ln\frac{R-t}{R+t} - t\ln\frac{R-x}{R+x}\right](x^2-t^2)^{-1} \quad \begin{pmatrix} 0 \leqslant t \leqslant a \\ 0 \leqslant x \leqslant a \end{pmatrix} \tag{2.19}$$

Into Eqs. (2. 18) and (2. 17) which were obtained we substitute Expression (2. 15) of function $F_{21}(r)$. As a result of this these equations are converted to the form

$$\varphi_1(t) = \int_0^a [K_{11}^{(1)}(t, x) + K_{11}^{(2)}(t, x)]\,\varphi_1(x)\,dx +$$

$$+ \int_a^R K_{12}^{(1)}(t, x)\,\varphi_2(x)\,dx + \chi_{12}(t) \qquad (0 \leqslant t \leqslant a) \tag{2.20}$$

$$\varphi_2(t) = \int_0^a K_{21}^{(1)}(t, x)\,\varphi_1(x)\,dx + \int_a^R K_{22}^{(1)}(t, x)\,\varphi_2(x)\,dx + \chi_{21}(t) \quad (a \leqslant t \leqslant R)$$

Here

$$K_{11}^{(2)}(t, x) = -\frac{4t}{\pi^2}\left[\frac{M_1(x, a)}{\sqrt{a^2-t^2}}\operatorname{arc\,tg}\left(\frac{R^2-a^2}{a^2-t^2}\right)^{1/2} +\right.$$

$$\left. + \int_a^R (r^2-t^2)^{-1/2}\operatorname{arc\,tg}\left(\frac{R^2-r^2}{r^2-t^2}\right)^{1/2}\frac{\partial M_1(x, r)}{\partial r}\,dr\right]$$

$$K_{12}^{(1)}(t, x) = \frac{4t}{\pi^2}\left[\frac{M_2(x, a)}{\sqrt{a^2-t^2}}\operatorname{arc\,tg}\left(\frac{R^2-a^2}{a^2-t^2}\right)^{1/2} +\right.$$

$$\left. + \int_a^R (r^2-t^2)^{-1/2}\operatorname{arc\,tg}\left(\frac{R^2-r^2}{r^2-t^2}\right)^{1/2}\frac{\partial M_2(x, r)}{\partial r}\,dr\right] \tag{2.21}$$

$$K_{21}^{(1)}(t, x) = -\frac{2tM_1(x, a)}{\pi\sqrt{t^2-a^2}} - \frac{2}{\pi}\int_a^t \frac{t}{\sqrt{t^2-r^2}}\frac{\partial M_1(x, r)}{\partial r}\,dr$$

$$K_{22}^{(1)}(t, x) = \frac{2tM_2(x, a)}{\pi\sqrt{t^2-a^2}} + \frac{2}{\pi}\int_a^t \frac{t}{\sqrt{t^2-r^2}}\frac{\partial M_2(x, r)}{\partial r}\,dr$$

$$\chi_{12}(t) = -\frac{2t}{\pi}\int_a^R \frac{\chi_{21}(\tau)\,d\tau}{t^2-\tau^2} + \chi_{11}(t), \quad \chi_{21}(t) = \frac{2}{\pi}\frac{d}{dt}\int_a^t \frac{rF_2(r)\,dr}{\sqrt{t^2-r^2}}$$

It is evident from Eq. (2. 20) that function $\varphi_1(t)$ indeed satisfies condition (2. 5).

Instead of function $\varphi_1(t)$ and $\varphi_2(t)$ we introduce new unknown functions $\varphi_{11}(t)$ and $\varphi_{21}(t)$, writing

$$\varphi_1(t) = \sqrt{a}\,(a^2 - t^2)^{-1/4}\varphi_{11}(t), \quad \varphi_2(t) = \sqrt{a}\,(t^2 - a^2)^{-1/4}\varphi_{21}(t) \qquad (2.22)$$

We further put

$$\begin{aligned} K_{11}(t, x) &= (a^2 - t^2)^{1/4}(a^2 - x^2)^{-1/4}[K_{11}^{(1)}(t, x) + K_{11}^{(2)}(t, x)] \\ K_{12}(t, x) &= (a^2 - t^2)^{1/4}(x^2 - a^2)^{-1/4}K_{12}^{(1)}(t, x) \\ K_{21}(t, x) &= (t^2 - a^2)^{1/4}(a^2 - x^2)^{-1/4}K_{21}^{(1)}(t, x) \\ K_{22}(t, x) &= (t^2 - a^2)^{1/4}(x^2 - a^2)^{-1/4}K_{22}^{(1)}(t, x) \end{aligned} \qquad (2.23)$$

Then Eqs. (2.20) are rewritten in the following manner:

$$\varphi_{11}(t) = \int_0^a K_{11}(t, x)\varphi_{11}(x)\,dx + \int_a^R K_{12}(t, x)\varphi_{21}(x)\,dx + \chi_1(t) \quad (0 \leqslant t \leqslant a) \qquad (2.24)$$

$$\varphi_{21}(t) = \int_0^a K_{21}(t, x)\varphi_{11}(x)\,dx + \int_a^R K_{22}(t, x)\varphi_{21}(x)\,dx + \chi_2(t) \quad (a \leqslant t \leqslant R)$$

On the segment $[0, R]$ we introduce functions

$$\varphi(t) = \begin{cases}\varphi_{11}(t) & (0 \leqslant t < a) \\ \varphi_{21}(t) & (a \leqslant t \leqslant R),\end{cases} \qquad \chi(t) = \begin{cases}\chi_1(t) & (0 \leqslant t < a) \\ \chi_2(t) & (a \leqslant t \leqslant R)\end{cases} \qquad (2.25)$$

and on the rectangle $0 \leqslant t \leqslant R,\ 0 \leqslant x \leqslant R$ the kernel $K(t, x)$, writing

$$K(t, x) = \begin{cases} K_{11}(t, x) & (0 \leqslant t < a, \quad 0 \leqslant x < a) \\ K_{12}(t, x) & (0 \leqslant t < a, \quad a \leqslant x \leqslant R) \\ K_{21}(t, x) & (a \leqslant t \leqslant R, \quad 0 \leqslant x < a) \\ K_{22}(t, x) & (a \leqslant t \leqslant R, \quad a \leqslant x \leqslant R) \end{cases} \qquad (2.26)$$

Then Eqs. (2.24) are written in the form of one integral equation

$$\varphi(t) = \int_0^R K(t, x)\varphi(x)\,dx + \chi(t) \quad (0 \leqslant t \leqslant R) \qquad (2.27)$$

It is easy to show that for conditions imposed above on function $g(\lambda)$ the kernel $K(t, x)$ is square-summable. Therefore, if function $F_1(r)$ satisfies conditions indicated above while $F_2(r)$ is such that function $\chi(t)$ is square-summable, then Eq. (2.27) will be a Fredholm integral equation of the second kind.

Substituting (2.10) into (2.4) we obtain

$$A(\lambda) = \int_0^a \varphi_1(t)\,[\sin\lambda t - tN(\lambda, t)]\,dt + \int_a^R \varphi_2(t)\cos\lambda t\,dt \qquad (2.28)$$

$$N(\lambda, t) = \frac{2}{\pi}\int_R^{+\infty}\frac{\cos\lambda\tau\,d\tau}{t^2 - \tau^2} \qquad \begin{pmatrix}0 \leqslant \lambda < +\infty \\ 0 \leqslant t \leqslant a\end{pmatrix} \qquad (2.29)$$

In this manner the solution of triple integral equations (2.1)–(2.3) is given by Formula (2.28) where functions $\varphi_1(t)$ and $\varphi_2(t)$ are expressed through the solution of integral equation of Fredholm with the aid of Eqs. (2.22) and (2.25). The kernel and the free term of integral equation (2.27) are given by (2.11), (2.19), (2.21), (2.23), (2.25) and (2.26).

Now let us examine triple integral equations of the second type

$$\begin{aligned}
&\int_{1}^{+\infty} \lambda^{3/2} A(\lambda) J_1(\lambda r)\, d\lambda = F_1(r) \qquad (0 \leqslant r < a) \\
&\int_{0}^{+\infty} \lambda^{3/2} A(\lambda) [1 - g(\lambda)] J_1(\lambda r)\, d\lambda = F_2(r) \qquad (a \leqslant r \leqslant R) \\
&\int_{0}^{+\infty} \lambda^{3/2} A(\lambda) J_1(\lambda r)\, d\lambda = 0 \qquad (R < r < +\infty)
\end{aligned} \tag{2.30}$$

Here $A(\lambda)$ is an unknown function. It is assumed that $g(\lambda)$ is continuous while function $\lambda^2 g(\lambda)$ is absolutely integrable on $[0, +\infty]$.

By means of analogous arguments it is established that the solution of these equations is given by Formula

$$A(\lambda) = \int_0^a [t^{1/2} J_{3/2}(\lambda t) + t^2 N_1(\lambda, t)]\, \varphi_1(t)\, dt + \int_a^R \varphi_2(t)\, \eta(\lambda, t)\, dt \tag{2.31}$$

Here

$$N_1(\lambda, t) = \frac{2}{\pi} \int_R^{+\infty} \frac{\eta(\lambda, \tau)\, d\tau}{t^2 - \tau^2} \quad \begin{pmatrix} 0 \leqslant \lambda < +\infty \\ 0 \leqslant t \leqslant a \end{pmatrix}$$

$$\begin{gathered}
\eta(\lambda, t) = \frac{1}{\lambda} [-t^{-3/2} J_{-1/2}(\lambda t) - t^{-1/2} \lambda J_{1/2}(\lambda t) + \sqrt{R}\, t^{-2} J_{-1/2}(\lambda R)] \\
\varphi_1(t) = a^{3/2} (a^2 - t^2)^{-1/4} \varphi_{11}(t), \qquad \varphi_2(t) = a^{1/2} (t^2 - a^2)^{-1/4} \varphi_{21}(t) \\
\varphi(t) = \begin{cases} \varphi_{11}(t) & (0 \leqslant t < a) \\ \varphi_{21}(t) & (a \leqslant t \leqslant R) \end{cases}
\end{gathered} \tag{2.32}$$

The function $\varphi(t)$ is found from the following integral equation:

$$\varphi(t) = \int_0^R K(t, \tau)\, \varphi(\tau)\, d\tau + \chi(t) \qquad (0 \leqslant t \leqslant R) \tag{2.33}$$

$$K(t, \tau) = \begin{cases} K_{11}(t, \tau) & (0 \leqslant t < a, \quad 0 \leqslant \tau < a) \\ K_{12}(t, \tau) & (0 \leqslant t < a, \quad a \leqslant \tau \leqslant R) \\ K_{21}(t, \tau) & (a \leqslant t \leqslant R, \quad 0 \leqslant \tau < a) \\ K_{22}(t, \tau) & (a \leqslant t \leqslant R, \quad a \leqslant \tau \leqslant R) \end{cases}$$

$$\begin{aligned}
K_{11}(t, \tau) = {}& \frac{(a^2 - t^2)^{1/4}}{(a^2 - \tau^2)^{1/4}} \left\{ \frac{1}{a} \left(\frac{2}{\pi}\right)^{3/2} \left[\sqrt{R^2 - a^2} + (a^2 - t^2)^{-1/2} t^2 \operatorname{arc\,tg} \left(\frac{R^2 - a^2}{r^2 - t^2}\right)^{1/2}\right] \times \right. \\
& \times M_3(\tau, a) + \left(\frac{2}{\pi}\right)^{3/2} \int_a^R \left[\sqrt{R^2 - r^2} + t^2 (r^2 - t^2)^{-1/2} \operatorname{arc\,tg} \left(\frac{R^2 - r^2}{a^2 - t^2}\right)^{1/2}\right] \times \\
& \times \frac{\partial}{\partial r} \left[\frac{1}{r} M_3(\tau, r)\right] dr + \frac{4\tau^2}{\pi^2} \left(\frac{1}{2(\tau^2 - t^2)} \left[\frac{1}{t} \ln \frac{R - t}{R + t} - \frac{1}{\tau} \ln \frac{R - \tau}{R + \tau}\right] + \right. \\
& \left.\left. + \frac{R}{2t\tau^2} \left[\frac{1}{R} - \frac{1}{2\tau} \ln \frac{R + \tau}{R - \tau}\right] \ln \frac{R + t}{R - t} \right) \right\}
\end{aligned}$$

$$K_{12}(t,\tau)=\frac{a\,(a^2-t^2)^{1/4}}{(\tau^2-a^2)^{1/4}}\left\{\left(\frac{2}{\pi}\right)^{3/2}\int\limits_a^R\left[\sqrt{R^2-r^2}+t^2\,(r^2-t^2)^{-1/2}\operatorname{arc\,tg}\left(\frac{R^2-r^2}{r^2-t^2}\right)^{1/2}\right]\times\right.$$

$$\times\frac{\partial}{\partial r}\left[\frac{1}{r}M_4(\tau,r)\right]'dr+\frac{R}{\pi t\tau^2}\ln\frac{R+t}{R-t}+\frac{1}{a}\left(\frac{2}{\pi}\right)^{3/2}\left[\sqrt{R^2-a^2}+\right.$$

$$\left.\left.+\,t^2\,(a^2-t^2)^{-1/2}\operatorname{arc\,tg}\left(\frac{R^2-a^2}{a^2-t^2}\right)^{1/2}\right]M_4(\tau,a)\right\}$$

$$K_{21}(t,\tau)=-\frac{(t^2-a^2)^{1/4}}{a(a^2-\tau^2)^{1/4}}\left(\frac{2}{\pi}\right)^{1/2}\left\{\frac{t^3M_3(\tau,a)}{a\sqrt{t^2-a^2}}+\right.$$

$$\left.+\,t^3\int\limits_a^t(t^2-r^2)^{-1/2}\frac{\partial}{\partial r}\left[\frac{1}{r}M_3(\tau,r)\right]dr\right\}$$

$$K_{22}(t,\tau)=-\left(\frac{2}{\pi}\right)^{1/2}t^3\,\frac{(t^2-a^2)^{1/4}}{(\tau^2-a^2)^{1/4}}\left\{\frac{M_4(\tau,a)}{a\sqrt{t^2-a^2}}+\right.$$

$$\left.+\int\limits_a^t(t^2-r^2)^{-1/2}\frac{\partial}{\partial r}\left[\frac{1}{r}M_4(\tau,r)\right]dr\right\}$$

$$M_3(\tau,r)=\int\limits_0^{+\infty}[\tau^{1/2}\lambda^{1/2}J_{3/2}(\lambda\tau)+\tau^2\lambda^{1/2}N_1(\lambda,\tau)]\,g(\lambda)\,J_1(\lambda r)\,d\lambda$$

$$M_4(\tau,r)=\int\limits_0^{+\infty}\lambda^{1/2}g(\lambda)\,\eta(\lambda,\tau)\,J_1(\lambda r)\,d\lambda$$

$$\chi(t)=\begin{cases}\chi_1(t) & (0\leqslant t<a)\\ \chi_2(t) & (a\leqslant t\leqslant R)\end{cases}$$

$$\chi_1(t)=\left(\frac{2}{\pi}\right)^{3/2}\frac{(a^2-t^2)^{1/4}}{a^{3/2}}\left\{\frac{1}{a}\left[\sqrt{R^2-a^2}+t^2\,(a^2-t^2)^{-1/2}\operatorname{arc\,tg}\left(\frac{R^2-a^2}{a^2-t^2}\right)^{1/2}\right]F_2(a)+\right.$$

$$+\int\limits_a^R\left[\sqrt{R^2-r^2}+t^2\,(r^2-t^2)^{-1/2}\operatorname{arc\,tg}\left(\frac{R^2-r^2}{r^2-t^2}\right)^{1/2}\right]\frac{\partial}{\partial r}\left[\frac{1}{r}F_2(r)\right]dr+$$

$$\left.+\,\frac{\pi}{2}\int\limits_0^t\frac{r^2F_1(r)\,dr}{t\sqrt{t^2-r^2}}\right\}\qquad(2.34)$$

$$\chi_2(t)=-\left(\frac{2}{\pi}\right)^{3/2}\frac{t^3\,(t^2-a^2)^{1/4}}{a^{3/2}}\left\{\frac{F_2(a)}{a\sqrt{t^2-a^2}}+\int\limits_a^t\frac{1}{\sqrt{t^2-r^2}}d\left[\frac{1}{r}F_2(r)\right]\right\}$$

We can show that the kernel $K(t,\tau)$ is square-summable. Functions $F_1(r)$ and $F_2(r)$ are assumed to be such that the free term $\chi(t)$ is square-summable. In addition to this it is assumed that the integral

$$\int\limits_0^t\frac{r^2F_1(r)\,dr}{\sqrt{t^2-r^2}}$$

becomes zero for $t=0$. Under these conditions (2.33) will be a Fredholm integral equation of the second kind.

3. Let us examine the infinite elastic layer. The region occupied by the layer is expressed in cylindrical coordinates in the following manner:

$$-l \leqslant z \leqslant l, \quad 0 \leqslant r < +\infty, \quad 0 \leqslant \varphi < 2\pi$$

The die which is ring-shaped in plan view and bounded by a surface of revolution is pressed with an axial force of magnitude P into the elastic layer which is in the temperature field (1.6). The layer is situated on a rigid smooth foundation. There is no friction between the die and the layer. The boundary conditions of the problem are written in the following manner:

$$\tau_{rz}(r, l) = 0; \quad \tau_{rz}(r, -l) = 0, \quad u_z(r, -l) = 0 \qquad (0 \leqslant r < +\infty) \quad (3.1)$$

$$\sigma_z(r, l) = 0 \qquad (0 \leqslant r < a, \quad R < r < +\infty) \quad (3.2)$$

$$u_z(r, l) = \psi(r) \qquad (a \leqslant r \leqslant R) \quad (3.3)$$

We are looking for a solution of the problem in the form of Eqs. (1.2), (1.4), (1.5) and (1.6). In this connection the unknown functions $A_1(\lambda), \ldots, A_6(\lambda)$ entering into (1.5) must be found from boundary conditions and from relationships (1.7). We can verify by direct substitution that functions (1.2) and (1.4) satisfy boundary conditions (3.1) and that relationships (1.7) are fulfilled, if functions $A_1(\lambda), \ldots, A_6(\lambda)$ have the following expressions:

$$\begin{aligned}
A_1(\lambda) &= \lambda^{-1}A_6(\lambda)(1 - 2\sigma - \lambda l \,\mathrm{th}\, \lambda l) - \gamma 2^{-1}\lambda^{-2}C_1(\lambda)\\
A_2(\lambda) &= \lambda^{-1}A_5(\lambda)(1 - 2\sigma - \lambda l \,\mathrm{cth}\, \lambda l) - \gamma 2^{-1}\lambda^{-2}C_2(\lambda)\\
A_3(\lambda) &= \lambda^{-1}A_6(\lambda)(1 - 2\sigma - \lambda l \,\mathrm{th}\, \lambda l) + \gamma 2^{-1}\lambda^{-2}C_1(\lambda) \qquad (3.4)\\
A_4(\lambda) &= \lambda^{-1}A_5(\lambda)(1 - 2\sigma - \lambda l \,\mathrm{cth}\, l\lambda) + \gamma 2^{-1}\lambda^{-2}C_2(\lambda)
\end{aligned}$$

$$A_5(\lambda) = \frac{[1 - g(\lambda)]}{\mathrm{sh}\,\lambda l} A(\lambda) - \frac{\gamma L_3(\lambda)}{4(\sigma - 1)\lambda L_1(\lambda) L_2(\lambda)}[C_1(\lambda) - C_2(\lambda)\,\mathrm{th}\,\lambda l]$$

$$A_6(\lambda) = \frac{2\,\mathrm{th}\,\lambda l}{L_1(\lambda) L_2(\lambda)} A(\lambda) + \frac{\gamma}{4(\sigma - 1)\lambda L_1(\lambda)}[C_1(\lambda) - C_2(\lambda)\,\mathrm{th}\,\lambda l]$$

$$L_2(\lambda) = \mathrm{ch}\,\lambda l + \lambda l / \mathrm{sh}\,\lambda l, \qquad L_3(\lambda) = \mathrm{sh}\,\lambda l - \lambda l / \mathrm{ch}\,\lambda l$$

$$L_1(\lambda) = 1 + \mathrm{th}\,\lambda l L_3(\lambda) / L_2(\lambda)$$

$$g(\lambda) = (\mathrm{ch}\,2\lambda l + 2\lambda l / \mathrm{sh}\,2\lambda l)^{-1}[(\mathrm{ch}\,2\lambda l + \mathrm{sh}\,2\lambda l)^{-1} + 2\lambda l / \mathrm{sh}\,2\lambda l] \quad (3.5)$$

where $A(\lambda)$ is a new unknown function; $C_1(\lambda)$ and $C_2(\lambda)$ are known functions of integral (1.6), γ is determined by Eq. (1.7).

Satisfying boundary conditions (3.2) and (3.3) we obtain triple integral equations

$$\begin{aligned}
&\int_1^{+\infty} \lambda A(\lambda) J_0(\lambda r)\, d\lambda = 0 \qquad (0 \leqslant r < a, \quad R < r < +\infty)\\
&\int_0^{+\infty} A(\lambda)[1 - g(\lambda)] J_0(\lambda r)\, d\lambda = F_2(r) \qquad (a \leqslant r \leqslant R)
\end{aligned} \qquad (3.6)$$

Here $g(\lambda)$ is given by Eq. (3.5)

$$F_2(r) = \psi(r) + \frac{\gamma}{4(\sigma - 1)} \int_0^{+\infty} \frac{\mathrm{sh}\,\lambda l}{\lambda} \Big\{C_2(\lambda) + \\ + \frac{L_3(\lambda)}{L_1(\lambda) L_2(\lambda)}[C_1(\lambda) - C_2(\lambda)\,\mathrm{th}\,\lambda l]\Big\} J_0(\lambda r)\, d\lambda. \qquad (3.7)$$

In this manner the solution of the formulated boundary value problem is given by Eqs. (1.2), (1.4), (1.5) and (1.6). Unknown functions $A_1(\lambda), \ldots, A_6(\lambda)$ entering into the integrals (1.5) are expressed through one unknown function $A(\lambda)$ with the aid of Eqs.

(3.4). For finding $A(\lambda)$ triple integral equations (3.6) are obtained which in form coincide with (2.1)–(2.3). Therefore their solution is given by Eq. (2.28) where $N(\lambda, t)$ is represented by (2.29), while functions $\varphi_1(t)$ and $\varphi_2(t)$ are expressed through the solution of the integral equation of Fredholm (2.27) with the aid of (2.22) and (2.25). We emphasize that in the present problem it is necessary to take into account in writing the expression for the kernel and the free term of the integral equation (2.27), that $F_1(r) = 0$, while the functions $g(\lambda)$ and $F_2(r)$ are given by (3.5) and (3.7).

Utilizing integrals (2.7) we find the formula for distribution of normal stresses under the die

$$\sigma_z(r, l) = \varepsilon(r) \qquad (a < r < R)$$

$$\varepsilon(r) = -\frac{G}{1-\sigma}\left\{\frac{1}{r}\frac{d}{dr}\int_r^R \frac{t\varphi_2(t)\,dt}{\sqrt{t^2-r^2}} +\right.$$

$$\left. + \frac{2}{\pi}\int_0^a \frac{\tau\varphi_1(\tau)}{(r^2-\tau^2)}\left[(r^2-\tau^2)^{-1/2}\operatorname{arctg}\left(\frac{R^2-r^2}{r^2-\tau^2}\right)^{1/2} - \frac{1}{\sqrt{R^2-r^2}}\right]d\tau\right\} \qquad (3.8)$$

4. Now let us examine the axisymmetric problem of pressure due to a die which is ring-shaped in plan view on the elastic half-space $z \geqslant 0$ located in the temperature field (1.9). There is no friction between the die and the half-space. The boundary conditions of the problem are

$$\tau_{rz}(r, 0) = 0 \qquad (0 \leqslant r < +\infty) \qquad (4.1)$$

$$\sigma_z(r, 0) = 0 \qquad (0 \leqslant r < a, \quad R < r < +\infty) \qquad (4.2)$$

$$u_z(r, 0) = \psi(r) \qquad (a \leqslant r \leqslant R) \qquad (4.3)$$

where $\psi(r)$ is a given function.

We are seeking the solution of the problem in the form of Eqs. (1.2), (1.4), (1.8) and (1.9). It is easy to verify that the boundary condition (4.1) and the relationship (1.10) are fulfilled if functions $A_1(\lambda)$, $A_2(\lambda)$ and $A(\lambda)$ are connected by the relationships

$$A_1(\lambda) = -\frac{1-2\sigma}{\lambda}A(\lambda) + \frac{\gamma}{2\lambda^2}C(\lambda), \qquad A_2(\lambda) = -\frac{1-2\sigma}{\lambda}A(\lambda) - \frac{\gamma}{2\lambda^2}C(\lambda)$$

Satisfying boundary conditions (4.2) and (4.3), we obtain the following triple integral equations for determination of function $A(\lambda)$:

$$\int_0^{+\infty} \lambda A(\lambda) J_0(\lambda r)\,d\lambda = 0 \qquad (0 \leqslant r < a, \; R < r < +\infty)$$

$$\int_0^{+\infty} A(\lambda) J_0(\lambda r)\,d\lambda = F_2(r) \qquad (a \leqslant r \leqslant R)$$

Here

$$F_2(r) = 2\psi(r) + \frac{\gamma}{4(1-\sigma)}\int_0^{+\infty}\frac{C(\lambda)}{\lambda}J_0(\lambda r)\,d\lambda$$

These equations are a particular case of the thriple equations (2.1), (2.2), (2.3). Therefore their solution is given by (2.28), where $N(\lambda, t)$ is given by (2.29) and functions $\varphi_1(t)$ and $\varphi_2(t)$ are given by

$$\varphi_1(t) = \sqrt{a}\,(a^2-t^2)^{-1/4}\varphi_{11}(t), \qquad \varphi_2(t) = \frac{2}{\pi}\frac{d}{dt}\int_a^t \frac{rF_2(r)\,dr}{\sqrt{t^2-r^2}}$$

Here the function $\varphi_{11}(t)$ is determined from the integral equation of Fredholm

$$\varphi_{11}(t) = \int_0^a K_{11}(t, x)\varphi_{11}(x)\,dx + \chi_1(t) \qquad (0 \leqslant t \leqslant a)$$

$$\chi_1(t)=\frac{4t}{\pi^2\sqrt{a}}(a^2-t^2)^{1/4}\left[F_2(a)(a^2-t^2)^{-1/4}\operatorname{arctg}\left(\frac{R^2-a^2}{a^2-t^2}\right)^{1/2}+\right.$$

$$\left.+\int_a^R (r^2-t^2)^{-1/4}F_2'(r)\operatorname{arctg}\left(\frac{R^2-r^2}{r^2-t^2}\right)^{1/2}dr\right]$$

$$K_{11}(t,x)=\frac{2}{\pi^2}\frac{(a^2-t^2)^{1/4}}{(a^2-x^2)^{1/4}}\left(x\ln\frac{R-t}{R+t^2}-t\ln\frac{R-x}{R+x}\right)(x^2-t^2)^{-1}\qquad\begin{pmatrix}0\leqslant t\leqslant a\\0<x\leqslant a\end{pmatrix}$$

The distribution law for normal stresses under the die will be

$$\sigma_z(r,0)=-2^{-1}\varepsilon(r)$$

where $\varepsilon(r)$ is given by (3.8).

5. Let us examine the problem of torsion of an elastic layer under the action of a rigid ring-shaped die coupled with it; it is required to find the function $v(r,z)$ which satisfies within the region $-l\leqslant z\leqslant l$, $0\leqslant r<+\infty$ the differential equation

$$\frac{\partial^2 v}{\partial z^2}+\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rv)\right]=0 \tag{5.1}$$

and on its surface the conditions

$$v(r,l)=\psi(r)\qquad(a\leqslant r\leqslant R) \tag{5.2}$$

$$\tau_{z\varphi}(r,l)=0\qquad(0<r<a,\quad R<r<+\infty) \tag{5.3}$$

$$v(r,-l)=0\qquad(0<r<+\infty) \tag{5.4}$$

Here $\psi(r)$ is a given function. The plane $z=-l$ is fixed and in the region $a\leqslant r\leqslant R$ of the plane $z=l$ the layer is subjected to torsion. The magnitude of the moment of external forces on this region is

$$M_k=2\pi\int_a^R r^2\tau_{z\varphi}(r,l)\,dr$$

It is easy to verify that the function

$$v=-\frac{1}{2}\int_0^{+\infty}\frac{\sqrt{\lambda}A(\lambda)}{\operatorname{cth}2\lambda l}\left[\frac{\operatorname{sh}\lambda z}{\operatorname{sh}\lambda l}+\frac{\operatorname{ch}\lambda z}{\operatorname{ch}\lambda l}\right]J_1(\lambda r)\,d\lambda \tag{5.5}$$

satisfies Eq. (5.1) and boundary condition (5.4), while the stress $\tau_{z\varphi}$ has the form

$$\tau_{z\varphi}=-\frac{G}{2}\int_0^{+\infty}\frac{\lambda^{3/2}A(\lambda)}{\operatorname{cth}2\lambda l}\left[\frac{\operatorname{ch}\lambda z}{\operatorname{sh}\lambda l}+\frac{\operatorname{sh}\lambda z}{\operatorname{ch}\lambda l}\right]J_1(\lambda r)\,d\lambda \tag{5.6}$$

Satisfying boundary conditions (5.3) and (5.4) we obtain triple integral equations for determination of unknown function $A(\lambda)$

$$\int_0^{+\infty}\lambda^{3/2}A(\lambda)J_1(\lambda r)\,d\lambda=0\qquad(0\leqslant r<a,\quad R<r<+\infty)$$

$$\int_0^{+\infty}\lambda^{1/2}A(\lambda)\left[1-g(\lambda)\right]J_1(\lambda r)\,d\lambda=F_2(r)\qquad(a\leqslant r\leqslant R)$$

Here

$$F(r)=-\psi(r),\quad g(\lambda)=[\operatorname{ch}2\lambda l(\operatorname{ch}2\lambda l+\operatorname{sh}2\lambda l)]^{-1} \tag{5.7}$$

These equations coincide with Eqs. (2.30) for $F_1(r) = 0$. Therefore their solution is given by Eqs. (2.31)–(2.33). In writing the kernel and the free term of the integral equation (2.33) it is necessary to take into account that in the given problem $F_1(r) = 0$, and functions $F_2(r)$ and $g(\lambda)$ are given by (5.7).

Utilizing integrals

$$\int_0^{+\infty} \lambda^{1/2} J_{1/2}(\lambda t) J_1(\lambda r)\, d\lambda = \begin{cases} (2t/\pi)^{1/2} r^{-1} (r^2 - t^2)^{-1/2} & (t < r) \\ 0 & (t > r) \end{cases}$$

$$\int_0^{+\infty} \lambda^{1/2} J_{-1/2}(\lambda t) J_1(\lambda r)\, d\lambda = \begin{cases} (2/\pi t)^{1/2} r^{-1} & (r > t) \\ (2/\pi t)^{1/2} r^{-1} [1 - t(t^2 - r^2)^{-1/2}] & (r < t) \end{cases}$$

and Expression (5.6), we find the distribution of stresses under the die

$$\tau_{z\varphi}(r, l) = e_1(r) \qquad (a < r < R)$$

where

$$e_1(r) = G\left(\frac{2}{\pi}\right)^{1/2} \frac{1}{r} \left\{ \frac{2}{\pi} \int_0^a \frac{\tau^2 \varphi_1(\tau)}{r^2 - \tau^2} \left[(r^2 - \tau^2)^{-1/2} \operatorname{arctg} \left(\frac{R^2 - r^2}{r^2 - \tau^2} \right)^{1/2} - \frac{1}{\sqrt{R^2 - r^2}} \right] d\tau + \right.$$

$$+ \frac{R}{\sqrt{R^2 - r^2}} \left[\frac{2}{\pi} \int_0^a \varphi_1(\tau) \left(\frac{1}{R} + \frac{1}{2\tau} \ln \frac{R - \tau}{R + \tau} \right) d\tau + \int_a^R \frac{\varphi_2(\tau)}{\tau^2}\, d\tau \right] -$$

$$\left. - \frac{1}{r} \frac{d}{dr} \int_r^R \frac{\tau \varphi_2(\tau)\, d\tau}{\sqrt{\tau^2 - r^2}} \right\}$$

6. In the case of torsion of the elastic half-space $z \geqslant 0$ under the action of a rigid die coupled with it, the boundary conditions of the problem are as follows:

$$v(r, 0) = \psi(r) \qquad (a \leqslant r \leqslant R)$$
$$\tau_{z\varphi}(r, 0) = 0 \qquad (0 \leqslant r < a,\ R < r < +\infty) \tag{6.1}$$

Here $\psi(r)$ is a given function. In the region $a \leqslant r \leqslant R$ of the boundary plane $z = 0$ the half-space is subjected to torsion. The magnitude of the moment of external forces on this region is

$$M_k = 2\pi \int_a^R r^2 \tau_{z\varphi}(r, 0)\, dr$$

We can verify that the function

$$v = -\int_0^{+\infty} \lambda^{1/2} A(\lambda) e^{-\lambda z} J_1(\lambda r)\, d\lambda \tag{6.2}$$

satisfies the differential equation (5.1). The stress $\tau_{z\varphi}$ corresponding to displacement (6.2) has the form

$$\tau_{z\varphi} = G \int_0^{+\infty} \lambda^{3/2} A(\lambda) e^{-\lambda z} J_1(\lambda r)\, d\lambda$$

Satisfying boundary conditions (6.1) we obtain triple integral equations for finding the unknown function $A(\lambda)$

$$\int_0^{+\infty} \lambda^{3/2} A(\lambda) J_1(\lambda r)\, d\lambda = 0 \qquad (0 \leqslant r < a,\ R < r < +\infty)$$

$$\int_0^{+\infty} \lambda^{1/2} A(\lambda) J_1(\lambda r)\, d\lambda = F_2(r) \qquad (a \leqslant r \leqslant R)$$

where

$$F_2(r) = -\psi(r) \tag{6.3}$$

These equations are a particular case of Eqs. (2.30). Therefore their solution is given by (2.31) in which functions $N_1(\lambda, t)$ and $\eta(\lambda, t)$ are written in terms of expressions (2.32) while functions $\varphi_1(t)$ and $\varphi_2(t)$ are given by

$$\varphi_1(t) = a^{1/2}(a^2 - t^2)^{-1/4}\varphi_{11}(t), \qquad \varphi_2(t) = a^{1/2}(t^2 - a^2)^{-1/4}\chi_2(t) \tag{6.4}$$

Here $\varphi_{11}(t)$ is found from the integral equation of Fredholm of the second kind

$$\varphi_{11}(t) = \int_0^a K_{11}(t, \tau)\varphi_{11}(\tau)\,d\tau + b(t) \qquad (0 \leqslant t \leqslant a)$$

$$K_{11}(t,\tau) = \frac{4}{\pi^2}\frac{\tau^2(a^2 - t^2)^{1/4}}{(a^2 - \tau^2)^{1/4}}\left\{\frac{1}{2(\tau^2 - t^2)}\left[\frac{1}{t}\ln\frac{R-t}{R+t} - \frac{1}{\tau}\ln\frac{R-\tau}{R+\tau}\right] + \right.$$
$$\left. + \frac{R}{2t\tau^2}\left[\frac{1}{R} - \frac{1}{2\tau}\ln\frac{R+\tau}{R-\tau}\right]\ln\frac{R+t}{R-t}\right\}$$

$$b(t) = \chi_1(t) + \int_a^R K_{12}(t,\tau)\chi_2(\tau)\,d\tau$$

$$K_{12}(t, \tau) = \frac{aR}{\pi t\tau^2}\frac{(a^2 - t^2)^{1/4}}{(\tau^2 - a^2)^{1/4}}\ln\frac{R+t}{R-t} \tag{6.5}$$

Functions $\chi_2(t)$ and $\chi_1(t)$ entering into (6.4) and (6.5) are written in terms of Expressions (2.34) in which $F_1(r) = 0$, while $F_2(r)$ is given by Eq. (6.3).

The distribution of stresses under the die is given by

$$\tau_{z\varphi}(r, 0) = -e_1(r) \qquad (a < r < R)$$

Here $e_1(r)$ is given by Eq. (5.8).

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